August 1977

Technical Report 10

RESEARCH INTO THE EXTENSION OF THE LIFE OF BANK NOTES: RESULTS OF 1973, 1975 and 1976 FIELD TRIALS

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Comments on this work would be welcome.

CONTENTS

ABSTRACT

In the three field trials conducted in 1973, 1975 and 1976, all the experimental bank notes showed a longer average circulating life than the regular two dollar notes of either the 1954 series or the new series.

The best results in extending the life of bank notes were obtained with 'Butvar' (polyvinylbutyral), the coating material in use for some years by European bank-note printers.

From the results of the field trials reported in this Technical Report, it is calculated that the use of this coating would effect a reduction of 15.5 percent in the number of bank notes that must be printed each year to maintain a fixed stock of notes in circulation. This reduction in printing requirements would defray the cost of coating.

RESEARCH INTO THE EXTENSION OF THE LIFE OF BANK NOTES

RESULTS OF 1973, 1975 AND 1976 FIELD TRIALS

1. Introduction

Since the inception of printing of Bank of Canada notes in 1935, there has been a continuing research programme into methods that could extend the circulation life of bank notes, such as improvements in the paper or the inks, and coating the printed notes with protective plastic.

Plastic coatings have been applied for a number of years by some European bank-note printers, for instance, in the Netherlands and Austria. That they continue in this practice would seem to indicate that it is of economic value.

Normal physical tests for such attributes as tear strength, bursting strength, elongation, fold resistance etc., do not furnish reliable evidence for estimating or predicting the mean life of notes. The sole reliable method for determining this is the field trial in which specially numbered control and experimental notes (80,000-100,000 of each) are put into normal circulation and a daily record is kept for a period of 12-16 months of the numbers of notes returned to the Bank as non-reissuable.

Two such field trials have been completed and a third is in progress. The results obtained from the three trials and their interpretation form the subject of this Report. The three trials are referred to by the year in which they were begun, viz 1973, 1975 and 1976.

2. 1973 Field Trial

2.1 Coating

For this trial, 95,000 two dollar Canadian notes of the 1954 series were coated by the printing works of the Banque Nationale de Belgique with 'Butvar', a trade name for polyvinylbutyral. The material was supplied by Hoechst A.G., West Germany, and was their grade MOWITAL B 30H. It was applied on both sides of the notes as an 11% w/v in a solution of 89 parts ethanol, 9 parts methanol and 2 parts acetone, by a Billhöfer coating machine to provide a final dry weight of 0.95 lbs per 3000 sq ft on each side.

2.2 Distribution of the experimental notes and counting of reject returns

These notes along with the same number of regular uncoated control notes were put into normal circulation in the Ottawa area over a period from 15 October to 15 November 1973 and the starting date of the trial was taken as 1 November 1973. Daily records of the numbers of notes returned as non-reissuable were kept by the Currency Division of the Bank of Canada for the succeeding 16 months, by which time the returns had dwindled to a trickle.

2.3 Discussion and interpretation of results

For both the control and the coated notes, the pattern of the rejections over time was similar. This can be seen in Figures 1 and 2 which show the numbers of rejects per month plotted against the time

Figure 1

1973 TRIAL - OTTAWA CONTROL REJECTS

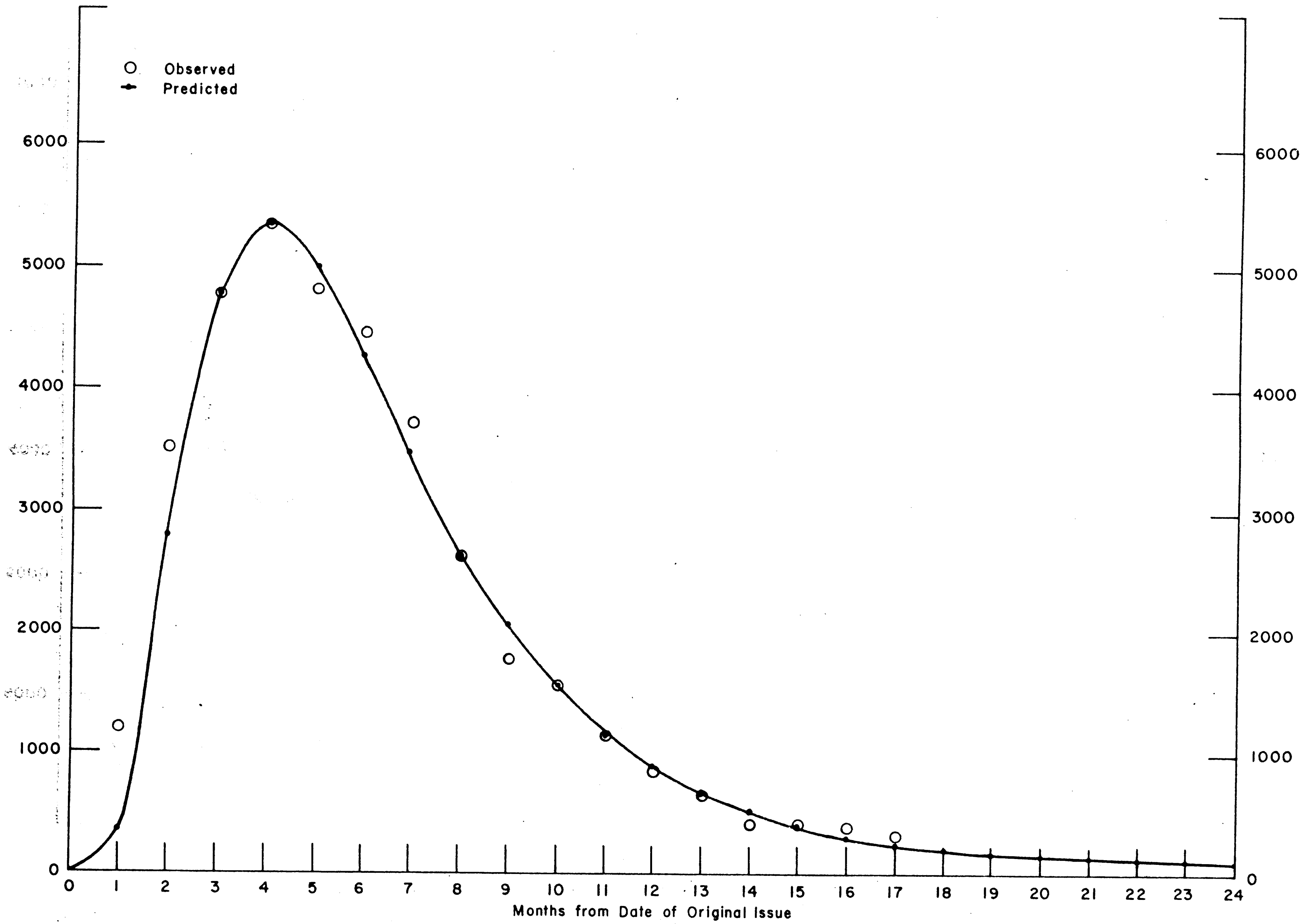
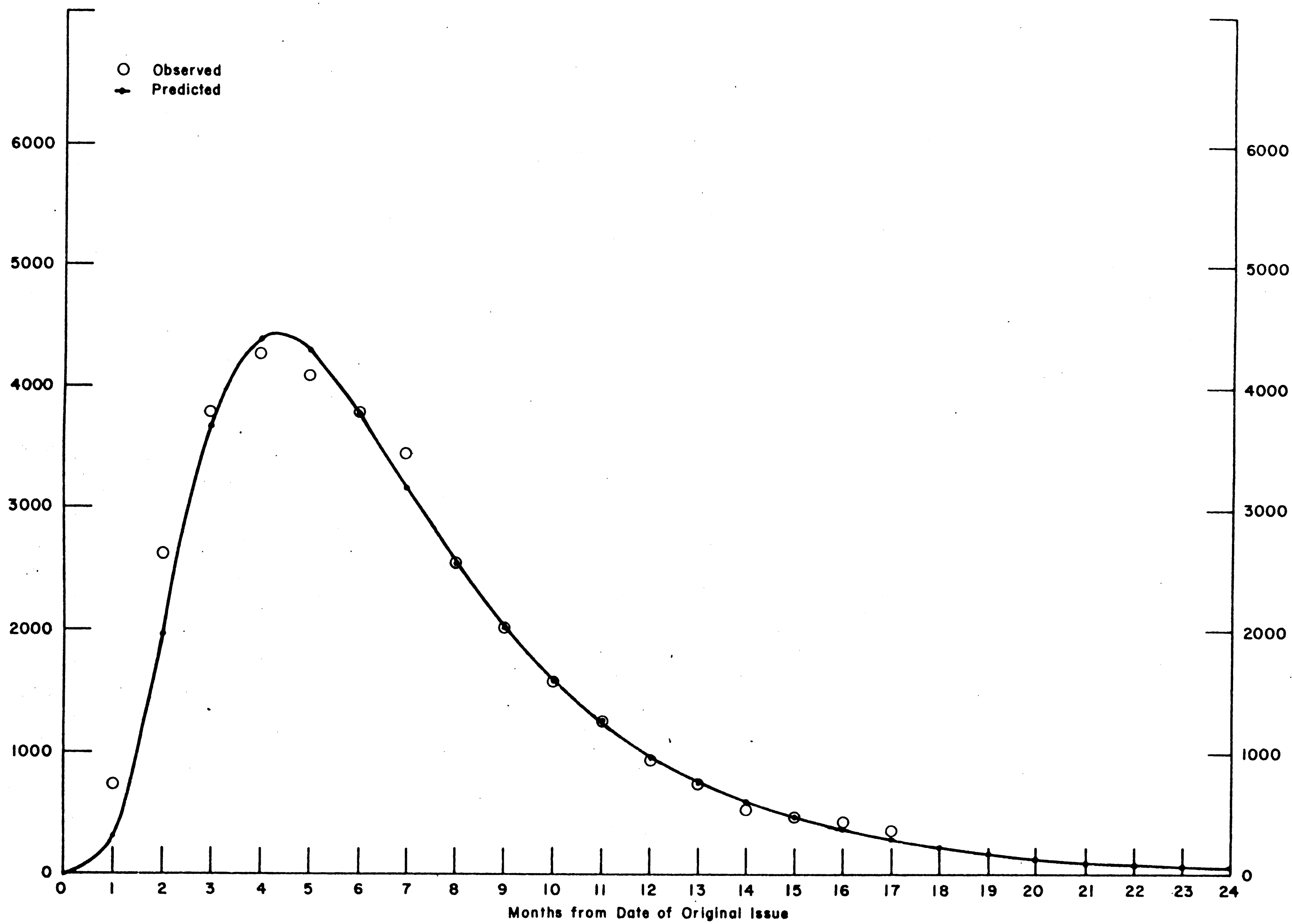


Figure 2

1973 TRIAL - OTTAWA 'BUTVAR' - COATED REJECTS



in months. The shape of the rejection curves appeared to approximate that of the lognormal distribution, and this was confirmed when the cumulative percentage of rejects was plotted against time on log-probability paper (see Figure 3). If distribution is lognormal, a straight-line relationship is observed. In this case, the relationship was linear over most of the range and the departure from linearity toward the end of the period was due to curtailment of the record keeping at 16 months.

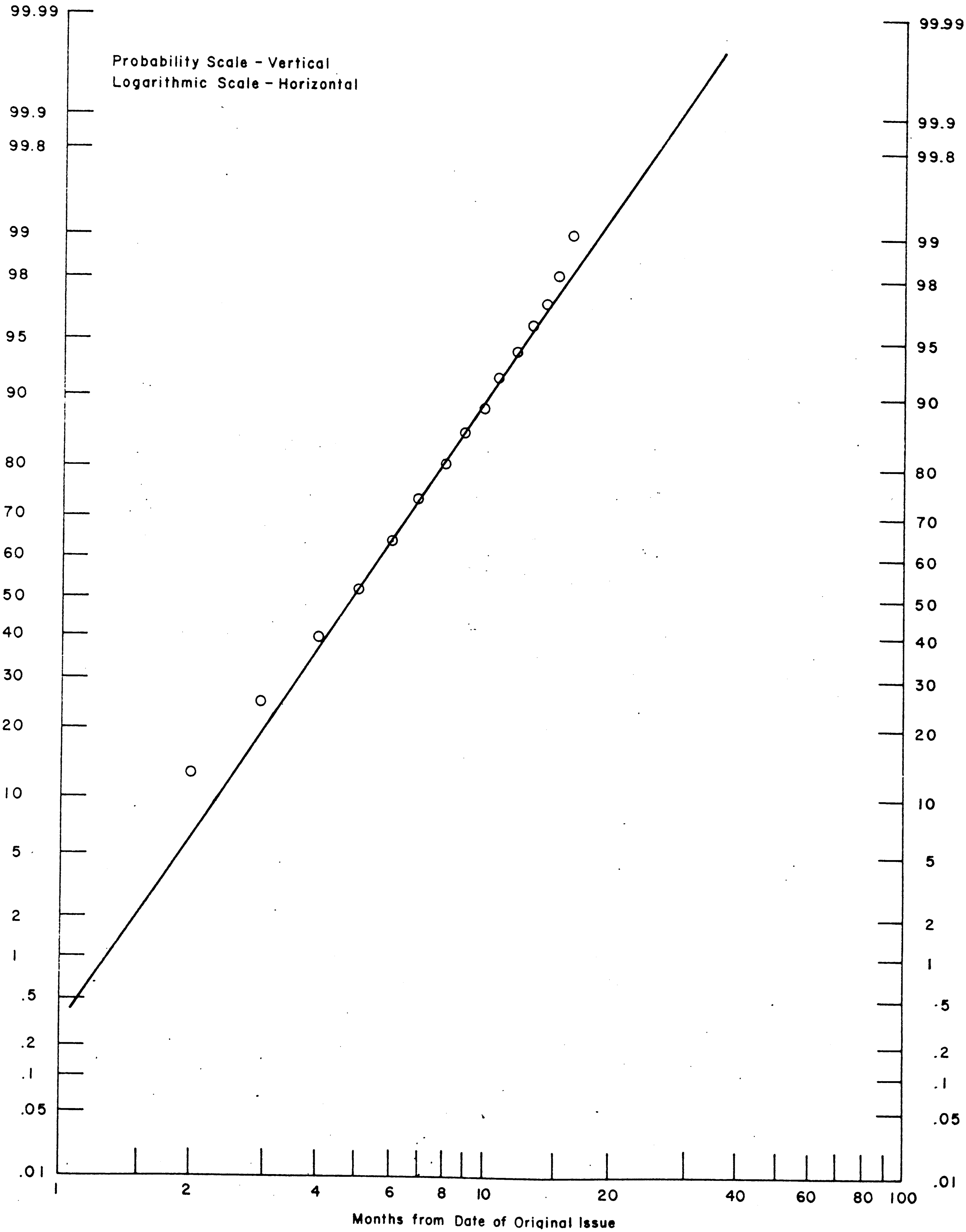
When the approximate nature of the distribution curve was thus determined, it was then possible to derive from the observed data the three parameters, f_0, σ and μ, that control and describe the lognormal distribution function (see Appendix I). This function fitted to the observed results could then be used to predict results well beyond the period of 12-13 months for which it would be practical and economical to count the number of notes rejected. The observed figures together with their extension calculated from the derived curve would provide a more accurate estimate of the mean note life.

Whether it was derived from observed data or from results calculated by means of the distribution curve, the mean note life was defined as the following simple ratio, which makes no assumptions as to the nature of the distribution:

$$\text{Mean note life(MNL)} = \frac{\text{Sum of the products of number of notes rejected per unit of time x the time since issue}}{\text{Sum of the notes rejected over the whole test period}}$$

Figure 3

1973 TRIAL - CUMULATIVE PERCENTAGE OF CONTROL REJECTS



The unit of time could be days, weeks, months or years, but for this test both days and 4-week months were the time units selected for calculating the mean note life.

In Table I the MNL estimates calculated solely from the observed results multiplied by the number of days since issue are compared with MNL results derived at monthly intervals from the fitted curve multiplied by the number of months since issue. From this comparison it can be seen that the estimates calculated by the two methods agree satisfactorily.

The results for a 24 month period (4-week months) of this trial are shown, along with the results of subsequent trials, in Table II. It can be seen that the 'Butvar'-coated notes in the 1973 test had a MNL 12.5 percent greater than the uncoated control notes.

From the observed results, it was noted that by the time the number of the returns had become very small, the total number of rejected notes that had been collected and counted in the Ottawa area still represented only 38 percent of the total 95,000 notes issued as control and only 35 percent of the 95,000 coated notes put into circulation. It was deduced that the reason for the low rate of returns was that a significant number of the notes had escaped from the Ottawa area to other areas such as Toronto and Montreal, and were thus not counted when they were sent back to the Bank's Agencies as non-reissuable.

A certain small number of notes probably filtered bank from the other areas into which they had escaped, and slightly increased the rate of returns for the Ottawa area toward the end of the counting

Table I

COMPARISON OF MEAN NOTE LIFE ESTIMATES OBTAINED DIRECTLY FROM OBSERVED RESULTS AND DERIVED FROM LOGNORMAL CURVE FOR 1973 'BUTVAR' RESULTS

Month	MNL from observed data in months	MNL from calculated data
1	1.0	1.0
2	1.7783	1.8541
3	2.4251	2.5254
4	3.0142	3.1124
5	3.5382	3.6341
6	4.0207	4.0993
7	4.4717	4.5139
9	5.1389	5.2113
10	5.4013	5.5025
11	5.6424	5.7607
12	5.8355	5.9892
13	6.0048	6.1913
14	6.1416	6.3699
15	6.2809	6.5277
16	6.4119	6.6671
17	6.5374	6.7902

Table II

1973 AND 1975 TRIAL RESULTS

	OTTAWA							SAINT JOHN			
	1973 Trial		1975 Trial					1975 Trial			
	Control	Butvar	Control	Butvar	N C	Paramel	F:R:L	Control	N C	Paramel	F:R:L
Mean note life in days at 24 months after issue, without re-placement	183.0	205.9	191.3	224.6	215.2	209.7	201.7	174.7	195.7	189.0	191.9
Percentage advantage over control		12.5		17.4	12.5	9.6	5.4		12.0	8.2	9.8
Average percent advantage in 1973 and 1975 trials				15.0							
Average percent advantage in Ottawa and Saint John trials					12.3	8.9	7.6				
Percentage advantage with re-placement		17.3* 19.0		29.9* 29.0	24.7*	22.3*	18.3*		16.7*	13.6*	7.4*
Average percent advantage in 1973 and 1975 trials				23.6* 24.0							
Average per-cent advantage in Ottawa and Saint John trials					20.7* 18.7	18.0* 17.1	12.9* 14.8				

Note abreviations: NC = Nitro-cellulose; F:R:L = 1/3 Flax: 1/3 Cotton Rag: 1/3 Linters
* Figures derived from the calculated results assuming lognormal distribution for the rejections.

period. Thus the truest estimate of the distribution of the rejects over time would be obtained by taking only results down to about 40 percent of the peak number of rejections.

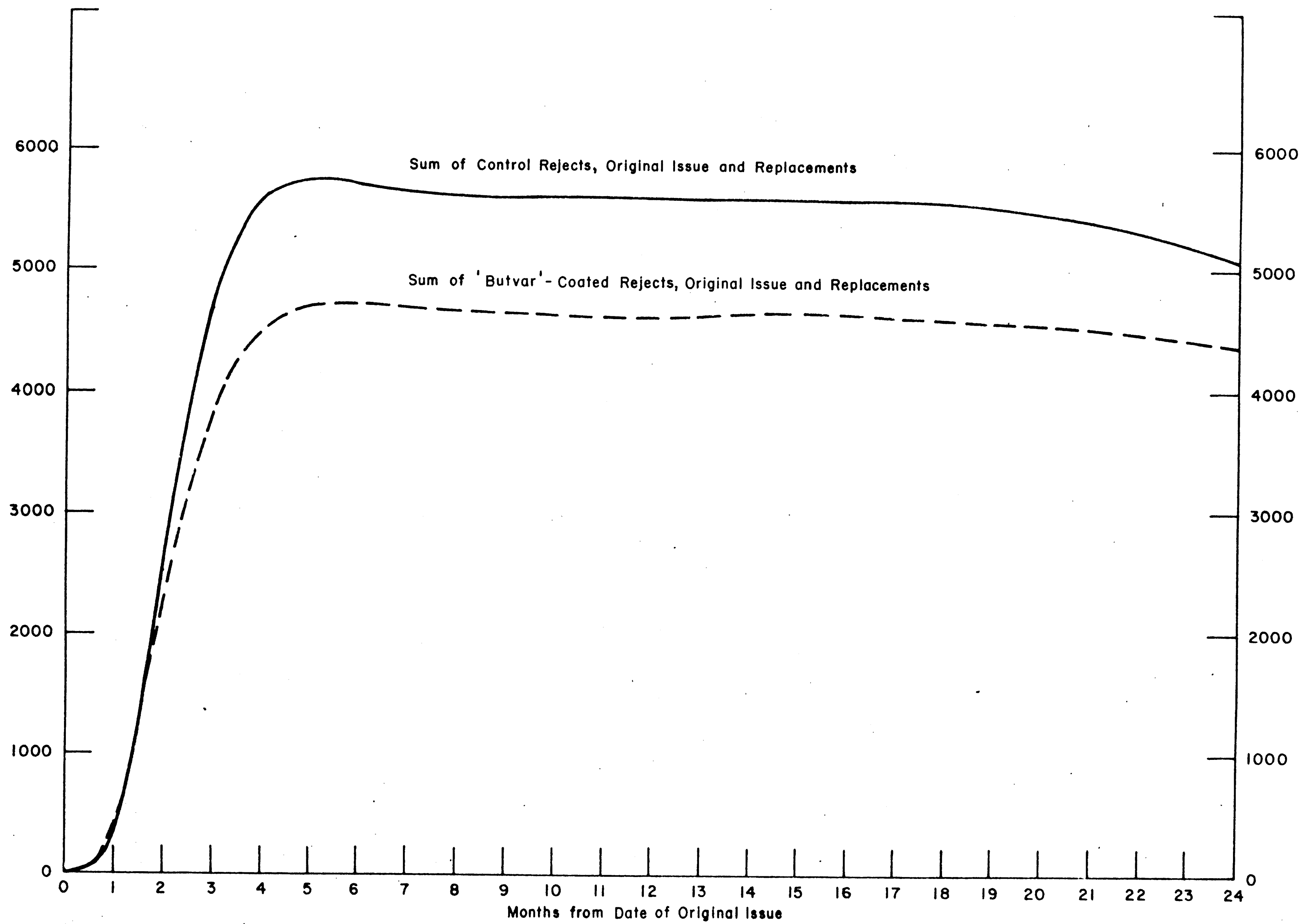
In all field trials, the observed figures indicated only the pattern of rejection without replacement. However for estimating the quantity of notes that normally need to be printed and issued, the Bank would require figures on rejection with replacement. Provided there is no adventitious increase or decrease in the demand for notes, the non-reissuable notes returned by the chartered banks are replaced by the Bank of Canada with an approximately equal number of new notes. These replacement notes in turn become worn out and are rejected in the same way as the original issue, as do also their further replacements.

If the number of notes in circulation is kept constant, that is, the bank notes are replaced as they are rejected, after a period of about 10 months from the original issue of a new series of notes an equilibrium would be established. From then on the number of notes rejected each month would become constant. By using the observed distribution of the rejected notes, it was possible to calculate the number of notes rejected with replacement for the first month or any desired number of months after the original issue. The way in which the sum of the rejection curves for the original issue and the replacements yielded, after a period, a constant number of rejections for the 1975 control note issues is illustrated in Figures 5 and 7.

Calculation of these equilibrium figures made it possible to estimate the number of notes that would have to be printed per annum

Figure 4

1973 TRIAL - OTTAWA REJECTS WITH REPLACEMENT



to replace those that would be rejected in the following year, provided the number of notes in circulation was held constant. From the results of the 1973 trial, such calculations were made for both the control notes and the 'Butvar'-coated notes. The resulting equilibrium rejection curves are presented graphically in Figure 4, with the corresponding data appearing in Table II. From these figures one can see that, on the basis of the 1973 field trial results, 19.0 percent fewer notes would have to be printed to maintain a constant number in circulation if the printed notes were coated with 'Butvar' plastic.

3. 1975 Field Trial

When the new series of Canadian two dollar notes was issued in 1975, it was considered advisable to repeat the trial. At the same time it was decided to take the opportunity to test another coating material and also two modifications of the paper on which the bank notes were printed.

3.1 Coating

Two lots of 80,000 notes each were coated on a Billhöfer coating machine, one lot with a formulation of the 'Butvar' plastic and the other with a nitro-cellulose plastic. This time however the coating was done in Canada. Transportation and security arrangements were made by staff of the Toronto Agency of the Bank. The 'Butvar' coating had a dry weight of 1.5 lbs per 3000 sq ft and the nitro-cellulose coating of 1.01 lbs per 3000 sq ft. The 'Butvar' was of the same grade and from the same supplier as that used in the 1973 test, viz MOWITAL B 30H from Hoechst A.G. Because of restrictions imposed by the Ontario government on the use of methanol however, the 'Butvar' for this test was dissolved in a solution consisting only of ethanol and acetone. The nitro-cellulose material was applied as a solution in toluene-xylene.

3.2 Modified paper

The two lots printed on modified paper consisted of:

(i) one 80,000 note lot in which the solid 'Parez' melamine-formaldehyde resin had been replaced in the manufacture of the paper by the improved liquid 'Paramel HE-2' resin acid colloid.

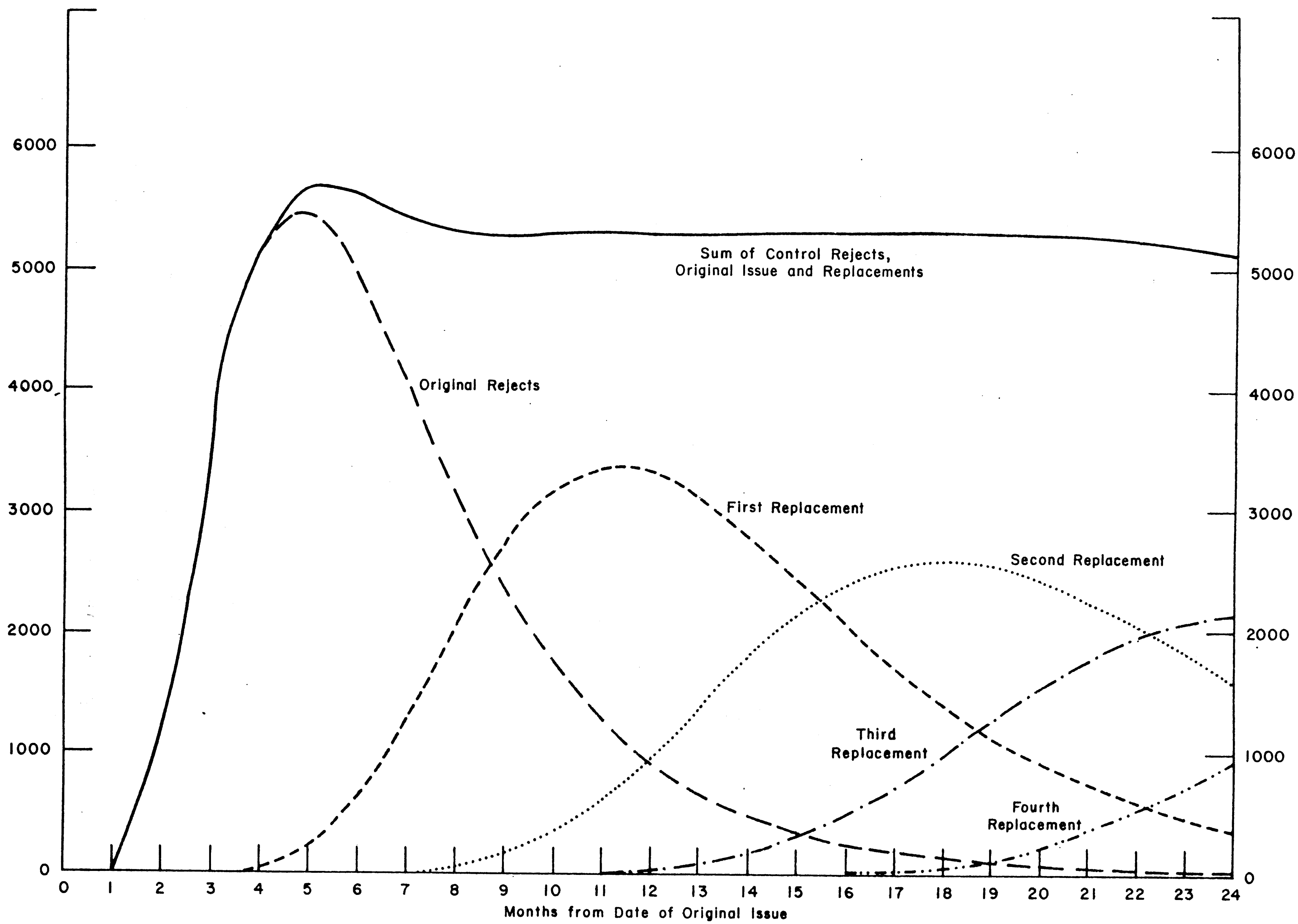
(ii) one 80,000 note lot printed on paper that had been made from 1/3 flax, 1/3 cotton rag and 1/3 cotton linters instead of the usual 1/4 flax and 3/4 cotton rag.

3.3 Distribution and counting

The experimental lots along with a control lot of 80,000 notes were issued simultaneously in the Ottawa and the Saint John, New Brunswick, areas on 17 November 1975. However, because of a miscalculation of the amount of coating material required, an insufficient quantity of 'Butvar'-coated notes was prepared. As a result none of the 'Butvar'-coated notes was issued through the Saint John Agency. All the trial notes considered non-reissuable were returned daily to the Currency Division of the Bank in Ottawa for

Figure 5

1975 TRIAL - OTTAWA CONTROL REJECTS WITH REPLACEMENT



sorting and counting.

3.4 Discussion and interpretation of results

The pattern of the rejections over time in the Ottawa area was similar to that of the 1973 trial (compare Figures 1 and 5) with the distributions again approximately lognormal. The distributions over time of the rejects from the control lots in Ottawa and Saint John are presented in Figures 5 and 7.

Although the rejection results from the four lots issued in the Saint John area again approximated the lognormal distribution (see Figure 7), they appeared markedly different from those from the Ottawa area in two respects. In Saint John the number of rejects at the peak was much higher, and for the same period of counting, the total number of rejected notes in each lot in the Saint John trial was a much higher proportion of the original issue of 80,000 notes.

This greater total number of rejects in the Saint John area compared with the Ottawa area was most likely due to the fact that the Saint John area is more closed than the Ottawa area. Therefore fewer notes escaped from the area and thus were not lost to the rejection count.

All the experimental notes had longer lives on the average than did the control groups as can be noted from Table II. The most marked increase, 17.4 percent, was shown by the 'Butvar'-coated notes. For the other types of experimental notes, the increase in MNL ranged from 5.4 percent to 12.5 percent for rejection <u>without</u> replacement.

When, as was done for the 1973 trial results, the equilibrium values for rejection with replacement were calculated for the 1975 trial results (see Table II and Figures 4 and 6), it was found that the 'Butvar'-coated notes showed the greatest advantage. With the 'Butvar' coating, 29.0 percent fewer notes would have to be printed per annum than would be required of the ordinary notes used as control. The other types of experimental notes would reduce the annual printing requirement by between 14.8 percent and 18.7 percent from that indicated by the control group.

The average of the improvements, indicated by the 1973 and 1975 trials in the Ottawa area in the case of the 'Butvar'-coated notes, and by the 1975 results in the Ottawa and the Saint John areas in the case of the other three types of experimental notes, are shown in Table II and may be summarized here.

Type of experimental note	Average percentage reduction in note requirements compared with controls: rejection with replacement
'Butvar'-coated	24.0
Nitro-cellulose-coated	18.7
'Paramel HE-2' resin incorporation	17.1
1/3 flax 1/3 cotton rag 1/3 cotton linters	14.8

No explanation has yet been found for the consistently longer average life for the bank notes issued in the Ottawa area than for those

Figure 6

1975 TRIAL - OTTAWA REJECTS WITH REPLACEMENT

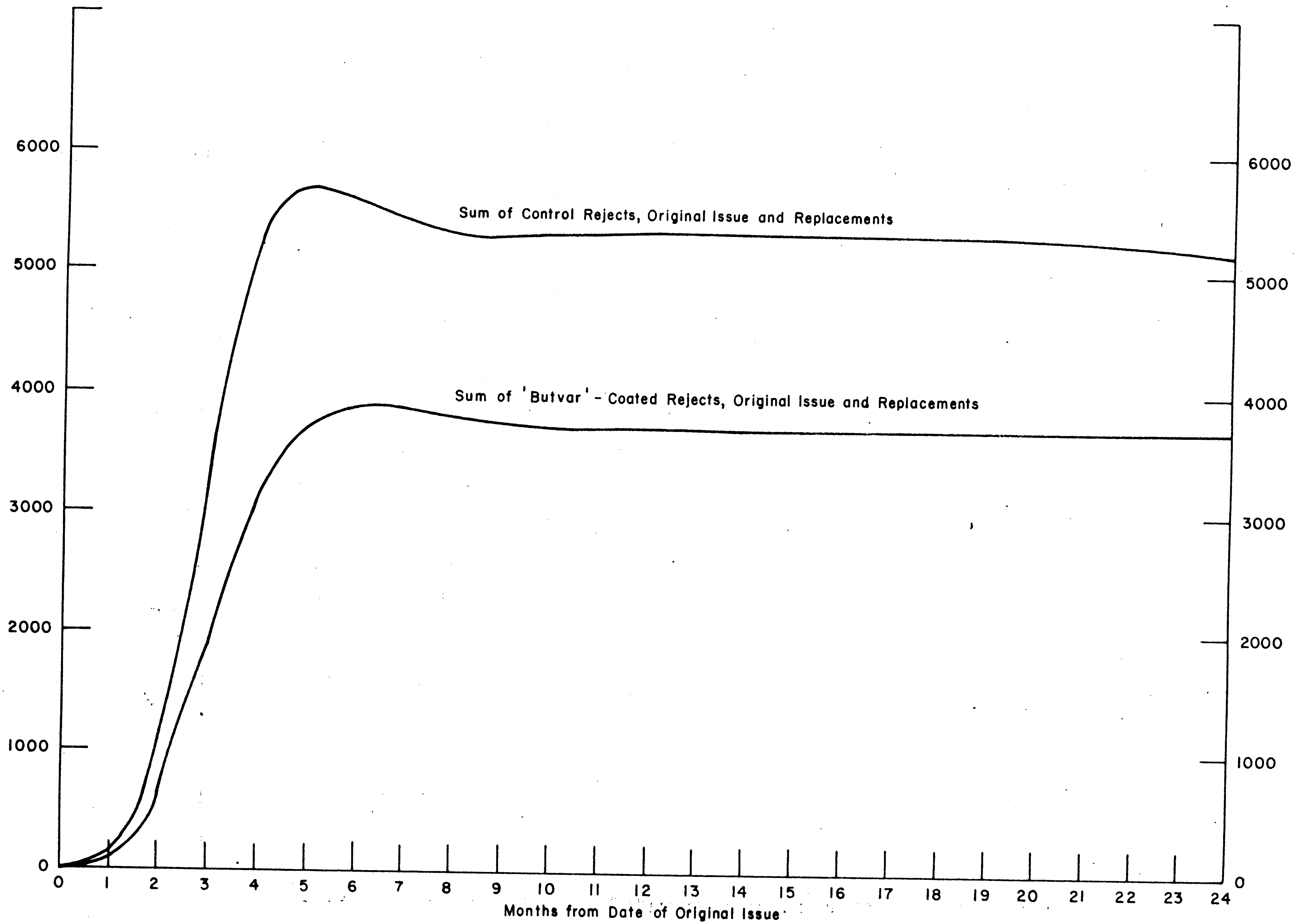
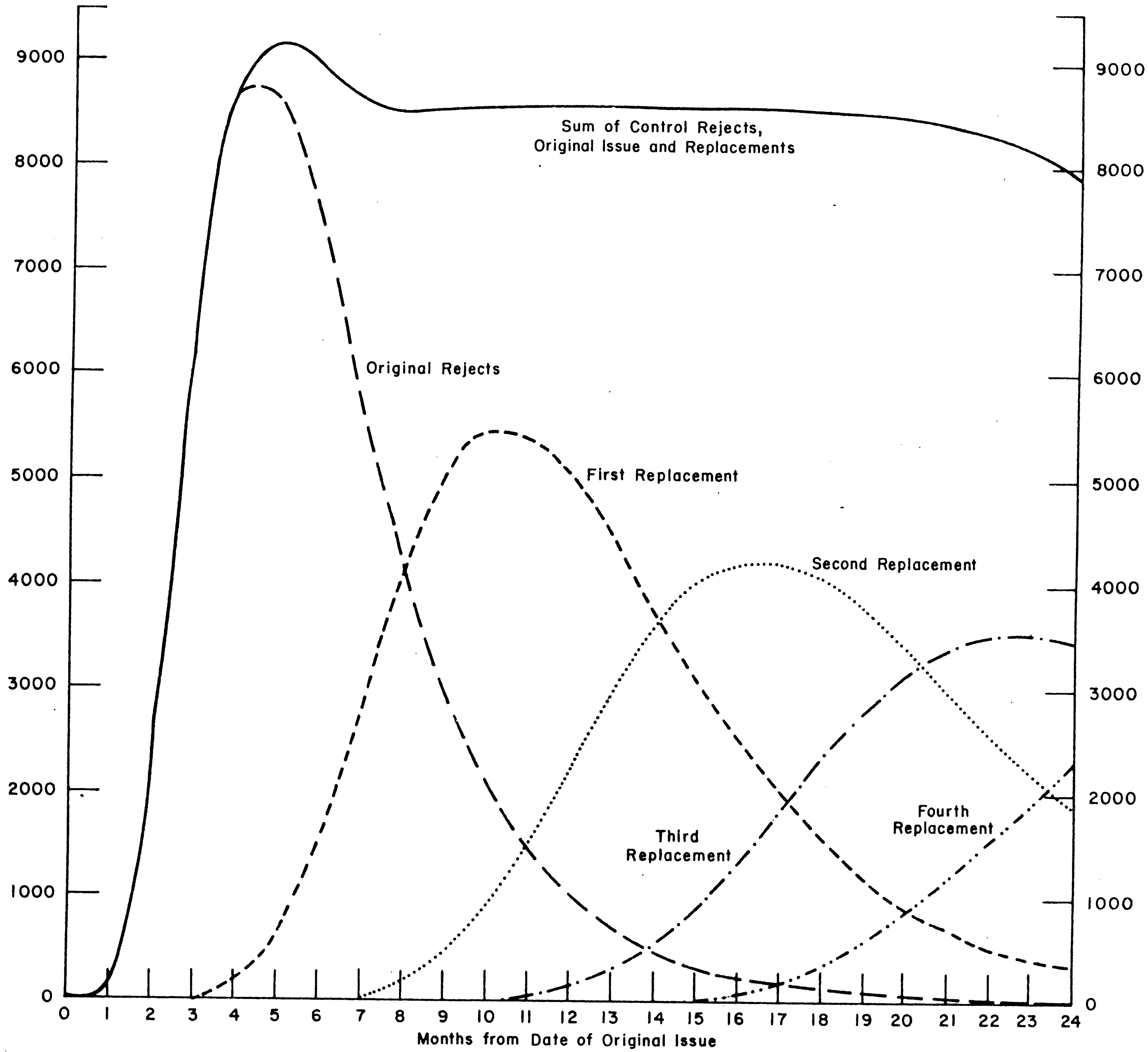


Figure 7

1975 TRIAL - SAINT JOHN CONTROL REJECTS WITH REPLACEMENT



issued in the Saint John area. The better results for the 'Butvar'-coated notes in the 1975 trial as compared with the 1973 trial were probably due to the heavier coating applied in the 1975 trial.

4. 1976 Field Trial

In September 1975 the suppliers of the bank-note paper altered its formulation by substituting the 'Paramel HE-2' acid colloid melamine resin for the 'Parez' melamine resin. In view of this change it was considered necessary to repeat the 1975 coating trials with two dollar notes printed on this new paper.

4.1 Coating

Two lots of 80,000 notes each were coated on a Billhöfer coating machine with the same formulations of the 'Butvar' and nitro-cellulose plastics as were used in the 1975 trial. Transportation and security arrangements were again made by staff of the Toronto Agency of the Bank of Canada. The 'Butvar' coating had a dry weight of 1.5 lbs per 3000 sq ft and the nitro-cellulose coating of 1.0 lbs per 3000 sq ft.

4.2 Distribution and counting

The two experimental 80,000 note lots along with a control lot were issued simultaneously in the Ottawa and the Saint John, New Brunswick, areas on 12 July 1976. All the trial notes that were considered non-reissuable were returned daily to the Currency Division of the Bank of Canada in Ottawa for sorting and counting.

4.3 Discussion and interpretation of results

The pattern of the rejections over time in both the Ottawa and Saint John areas closely resembled that of the 1975 trial. Again the distributions were approximately lognormal.

Both types of experimental notes had, on the average, longer lives than the control notes. The largest increase in the note life was again shown by the 'Butvar'-coated notes, although the increases for both 'Butvar' and nitro-cellulose were smaller than in the 1975 trial. This reduction in the advantage of the coated over the uncoated notes can be explained by the improvement in the paper that resulted from the substitution of the 'Paramel HE-2' resin for the 'Parez' resin. That such an improvement could be expected had already been indicated by the 1975 trial in which the mean note life of the 'Paramel HE-2' notes was increased by 9.6 percent in Ottawa and 8.2 percent in the Saint John area over that of the 'Parez' control notes, and the average advantage with replacement was 17.1 percent.

In the 1975 Ottawa trial the mean note life of the 'Butvar'-coated notes was 7.1 percent greater than that of 'Paramel HE-2' notes and the corresponding advantage with replacement was 5.9 percent. No final comparable figures are yet available from the results of the 1976 trial.

4.4 Statistical comments on the results

The findings of the initial report were checked by Dr. R. Haas of the Banking and Finance Department of the Bank of Canada. His comments are included as Appendix III.

From these comments it will be noted that the precision of the determination of all estimates of the mean note life was considered satisfactory but that the reject distribution function cannot be said to be closely lognormal.

In view of this latter comment, the ratio of the number of coated to uncoated notes, taking into account replacement, was recalculated using the observed distribution of rejects, that is, making no assumptions as to the mathematical nature of their distribution. The percentage increases in note life so calculated are shown in Table II along with those calculated from the lognormal distribution curve.

For non-normal (non-Gaussian) distributions, the distribution-free Kolmogorov-Smirnov statistic [2, 6, 7, 8, 9, 10] is preferred to the chi-squared test for goodness of fit used in Appendix III. Application of the Kolmogorov-Smirnov test for goodness of fit indicated that although the observed distributions were not lognormal within a 95 percent confidence interval, they did approach it and could be regarded as close to lognormal.

There is another distribution, that of Weibull [11, 3], which in certain ranges closely resembles the lognormal. It was therefore considered advisable to determine whether the observed distributions could be fitted by the Weibull distribution, which has the following formulation:

Probability density function (p.d.f.)

$$f_w\,(t;\sigma,\lambda) = \frac{\lambda}{\sigma}\left(\frac{t}{\sigma}\right)^{\lambda-1} \exp\left\{-\left(\frac{t}{\sigma}\right)^{\lambda}\right\}$$

Table III

NUMBER OF DAYS FOR REJECTION OF 26,000 NOTES

Trial	OTTAWA				SAINT JOHN			
	Control	Paramel	Butvar	NC	Control	Paramel	Butvar	NC
1973	182	-	234	-	-	-	-	-
1975	212	301	364	301	137	150	-	157
1976	322	-	364	332	158	-	176	185

Table IV

NUMBER OF NOTES REJECTED IN 364 DAYS

Trial	OTTAWA				SAINT JOHN			
	Control	Paramel	Butvar	NC	Control	Paramel	Butvar	NC
1973	36,492	-	32,034	-	-	-	-	-
1975	33,386	27,911	26,037	27,944	51,054	47,406	-	47,084
1976	27,204	-	26,013	26,961	43,045	-	40,586	40,315

General cumulative distribution function (c.d.f.) $F_w(t;\sigma,\lambda) = 1 - \exp\{-(\frac{t}{\sigma})^{\lambda}\}$

In order to discriminate between the lognormal and the Weibull functions, use is made of their hazard functions, which are markedly different.

$$h(t) = \frac{p.d.f.}{1 - (c.d.f.)}$$

The hazard function can be calculated from the observed p.d.f. and c.d.f. and the resultant curve compared with those derived from the lognormal p.d.f. and c.d.f. and the Weibull p.d.f. and c.d.f. The lognormal hazard function curve resembles the lognormal distribution curve; it rises steeply to a peak, descends and flattens out, while the Weibull hazard function does not display this behaviour.

Empirical hazard functions were calculated for the five control lots of notes issued in the 1973, 1975 and 1976 trials. All the resulting curves were of the lognormal hazard function type and thus furnished strong evidence that the observed frequency distributions were of the lognormal type.

There are two other indicators for the life of the notes which, like the method used in section 2.3 for calculating the mean note life, make no assumptions as to the distribution of the rejects with time. The first consists of comparing the number of days required for the rejection of the same number of control and coated notes. The selected number of notes should be as large as possible and the test is meaningful only when at least one third of the notes issued have been rejected. Such a comparison for the rejection of 26,000 notes in the three trials appears in Table III.

It should be remembered that the 1975 'Paramel' notes and the 1976 control notes were essentially the same and have in fact given similar results. The significant extension of the life of notes made from the new paper containing 'Paramel HE-2' acid colloid melamine resin compared with notes previously printed on paper containing 'Parez' melamine resin was clearly demonstrated.

The second indicator, in some respects the converse of the first, consists of a comparison of the number of notes rejected after a certain specified time, which should be as long as possible. Because the 1976 trial is not sufficiently advanced at the date of this report, the time used in the comparison must be somewhat shorter than is desirable; this deficiency should be taken into account in a study of the results shown in Table IV.

Again the superiority of the 'Paramel' paper over the 'Parez' paper was clearly shown, and the 1975 results with 'Paramel' paper were similar to the 1976 results given by the control lot.

As a general observation from Tables III and IV, the increase in the life of the control, ie ordinary, notes is significant. The number of days for the rejection of 26,000 notes increased by 77 percent between 1973 and 1976, and the number of notes rejected in 364 days declined by 25.5 percent between 1973 and 1976.

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ACKNOWLEDGEMENTS

The execution of the three trials has involved many groups and people, and my indebtedness for their assistance and co-operation is acknowledged to the following:

Banque Nationale de Belgique Printing Works for the coating in the 1973 Trial.

Vivian Associates, Toronto, for the coating in the 1975 Trial.

A.E. Long Ltd., Toronto, for the coating in the 1976 Trial.

Pierce and Stevens (Canada) Ltd., Fort Erie, for supplying the coating materials gratis for the 1975 and 1976 Trials, and for very helpful advice and guidance on coating materials and related problems.

The staff of the Toronto Agency of the Bank for transportation and security arrangements in the 1975 and 1976 Trials.

The staff of the Saint John Agency for distribution and return arrangements for the last two trials.

The staff of the Currency Division of the Bank for distribution arrangements in the Ottawa area and for recording non-reissuable returns in all three trials.

The Computer Services Division of the Bank for assistance in processing the accumulated data.

Appendix I

DETERMINATION OF THE PARAMETERS FOR THE ELEVEN LOGNORMAL DISTRIBUTION FUNCTIONS

The general frequency distribution function of the lognormal distribution is:

$$f(t) = \frac{f_0}{t\sigma\sqrt{2\pi}} \exp[-\frac{1}{2\sigma^2}\{\ln(t) - \mu\}^2] \qquad (1)$$

where $f(t)$ is the number of notes rejected during the t th month

f_0 is the total number of notes rejected

t is time in 4-week months

σ is a parameter related to the coefficient of variation of the distribution, such that $\sigma^2 = \ln(1+\eta^2)$ where η is the coefficient of variation

$\ln(t)$ is the natural or Napierian logarithm of t

μ is a parameter related to the mean α of the distribution, such that $\mu = \ln\alpha - 1/2\sigma^2$

In this study, there are available 10 to 12 values of f(t) and the corresponding values of t, for the determination of the three parameters f_0, σ and μ, which is performed as follows:

Selecting three values of f(t) and the corresponding values for t, transferring the t's to the left hand side of equation (1) and taking the ratio of the two expressions, I obtain the equation:

$$\frac{t_1 f(t_1)}{t_2 f(t_2)} = \frac{\exp[-(1/2\sigma^2)\{\ln(t_1)-\mu\}^2]}{\exp[-(1/2\sigma^2)\{\ln(t_2)-\mu\}^2]} = G \qquad (2)$$

where all the values on the left hand side are known, and f_0 and $\sigma\sqrt{2\pi}$ have cancelled out. This equation is designated as G.

Similar equations are obtained for the pairing of $f(t_1)$, t_1 with $f(t_3)$, t_3 and $f(t_2)$, t_2 with $f(t_3)$, t_3.

Taking logarithms of both sides of equation (2), I obtain:

$$2(\ln G) = -\frac{[\{\ln(t_1) - \mu\}^2 - \{\ln(t_2) - \mu\}^2]}{\sigma^2} \qquad (3)$$

and similar equations for the other two pairings.

For simplification of and ease of handling the equations, put $\ln(t_1) = p$, $\ln(t_2) = q$ and $\ln(t_3) = r$.

Then $2(\ln G) = + \{(q-\mu)^2 - (p-\mu)^2\}/\sigma^2$ (3a)

Again when ratios of pairs from the three equations are taken, σ^2 is cancelled out and there is obtained:

$$\frac{2(\ln G)}{2(\ln G^1)} = \frac{(a-\mu)^2 - (p-\mu)^2}{(r-\mu)^2 - (p-\mu)^2} = H \qquad (4)$$

which is an equation where μ is the only unknown, and which is solved by the following reductions:

$$H\{(r-\mu)^2 - (p-\mu)^2\} = (q-\mu)^2 - (p-\mu)^2$$

μ^2 cancels out.

$$Hr^2 - 2\mu Hr - Hp^2 + 2\mu Hp + p^2 - 2\mu p - q^2 + 2\mu q = 0$$

$$-2\mu Hr + 2\mu Hp - 2\mu p + 2\mu q = q^2 + (H-1)p^2 - Hr^2$$

$$2\mu\{(H-1)p + q - Hr\} = (H-1)p^2 + q^2 - Hr^2$$

Therefore

$$\mu = \frac{(H-1)p^2 + q^2 - Hr^2}{2\{(H-1)p + q - Hr\}} \qquad (5)$$

From equation (3a)

$$\sigma^2 = (q-\mu)^2 - (p-\mu)^2 /2(\ln G)$$

and from equation (1)

$$f_0 = \{f(t)t\sigma\sqrt{2\pi}\} \exp[\frac{1}{2\sigma^2}\{\ln(t) - \mu\}^2].$$

Appendix II

THE LOGNORMAL DISTRIBUTION

The probability that a note will be rejected as non-reissuable after a certain time is given by an error function where the independent variable is not time, but the logarithm of the time.

This result, which at first sight may appear curious, would seem to result from the fact that in rejection we are, in effect, performing a classification process. The suggestion that this interpretation is approximately true is given by the following quotation from the monograph "The Lognormal Distribution", by J. Aitchison and J.A.C. Brown [1].

"The central idea of the theory of breakage may be carried over into a theory of classification. It is a curious fact that when a large number of items is classified on some homogeneity principle, the variate defined as the number of items in a class is often approximately lognormal. Examples of this phenomena we have noted are the number of persons in a census occupation class, the number of Sino-Japanese characters in a lexicographical group, and the outlay by households on classes of commodities. At first sight it may appear that, in classification problems of this type, the classifier is free to produce any distribution he chooses. But in practice, for a meaningful classification, some principle of homogeneity must be followed, and we suggest that the application of such a principle may lead to a process closely analogous to the breakage process described above. Thus a detailed list of occupations, collected from census returns, may first be divided into manual and non-manual, then each of these into skilled and unskilled, and so on."

Appendix III

A STATISTICAL EXAMINATION OF THE 1973 AND 1975 'EXTENSION OF LIFE' FIELD TRIALS

Two aspects of the 21 October 1976 field report are examined in this note. The first question concerns the extension to note life from a variety of treatments. Briefly, both the 1973 and 1975 data lead us to conclude the most effective treatment, that is, 'Butvar' coating, extends average note life by two weeks plus or minus a small error term. The second aspect of this report is a test of the validity of the lognormal distribution function assumption. If we know the manner in which the life of bank notes is distributed, then we can make further statements concerning such things as necessary rates of replacement to maintain constant stocks of notes. It has been suggested that the lognormal distribution is a good description of the pattern of bank-note rejections. Unfortunately the statistical test we used does not allow us to accept this hypothesis.

Estimates of mean note life

It is not necessary to know the distribution properties of statistical populations in order to make inferences about the difference between two means. The central limit theorem assures us that as long as we have a 'large' sample, the test we used will yield accurate results. The sample sizes we have are more than ample to satisfy the necessary condition.

The results of the tests are given in the Appendix III Table below. The level of confidence column tells us how often we would expect the true difference between the experimental mean and control population mean to be outside the estimated range, ie, .05 is 5 chances in 100, .01 is 1 chance in 100, and .005 is 5 chances in 1,000.

Two things are immediately obvious from the Table. First, 'Butvar' is the most effective treatment and the percentage increase in note life in both experiments is very nearly the same (8.58% $\pm$ .85% in 1973 and 8.44% $\pm$ .66% in 1975, both at the .05 confidence level). The second striking feature is the difference between the 1975 test results in Saint John and Ottawa. Especially perplexing is the different ordering of the results that show for instance that PARAMEL HE-2 is the most inefficient coating in Saint John but not in Ottawa. We have no explanation for this.

Testing the lognormal assumption

If note life, NL, is lognormally distributed, the relative frequency of a given NL is described by

$$P(NL) = \frac{1}{NL\sigma\sqrt{2\pi}}\, e - 1/2 \left(\frac{\log NL - \mu}{\sigma}\right)^2$$

where μ and σ^2 are the mean and variance of the underlying normal distribution of X where $X = \log_e NL$. We can test the validity of the assumption of lognormality by testing to see if X is normally distributed. This was done by means of a chi-squared test. The calculated chi-squared statistic was well above the level that would

Appendix III Table

ESTIMATED DIFFERENCE IN POPULATION MEANS

Mean of experimental group less control mean	Difference in months	Difference as a percent of control	Level of confidence
1973 - Ottawa			
'BUTVAR'-CONTROL	.51653±.0513	8.58%± .85%	.05
	±.0675	±1.12%	.01
	±.0736	±1.22%	.005
1975 - Ottawa			
'BUTVAR'-CONTROL	.49857±.0389	8.44%± .66%	.05
	±.0511	± .86%	.01
	±.0557	± .94%	.005
NITRO-CELLULOSE-CONTROL	.40422±.0376	6.84%± .64%	.05
	±.0494	± .84%	.01
	±.0539	± .91%	.005
PARAMEL HE-2-CONTROL	.3776 ±.0373	6.39%± .63%	.05
	±.0490	± .83%	.01
	±.0535	± .91%	.005
1/3 FLAX, RAG, LINTERS-CONTROL	.16383±.0378	2.77%± .64%	.05
	±.0497	±..84%	.01
	±.0543	± .92%	.005
1975 - Saint John			
NITRO-CELLULOSE-CONTROL	.24962±.0274	4.61%± .51%	.05
	±.0360	± .67%	.01
	±.0393	± .73%	.005
1/3 FLAX, RAG, LINTERS-CONTROL	.21274±.0270	3.93%± .50%	.05
	±.0355	± .66%	.01
	±.0327	± .72%	.005
PARAMEL HE-2-CONTROL	.20145±.0272	3.72%± .50%	.05
	±.0357	± .66%	.01
	±.0390	± .72%	.005

have allowed us to accept the lognormality assumption. Thus while we can make statements about significant differences between the control mean and the experimental mean, we cannot make any statements about the distribution of the life of bank notes.

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November 9, 1976